Data-Driven Multi-Stage Stochastic Optimization on Time Series

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Joint work with: Nam Ho-Nguyen (Univ. of Sydney),
Jim Luedtke (UW-Madison)

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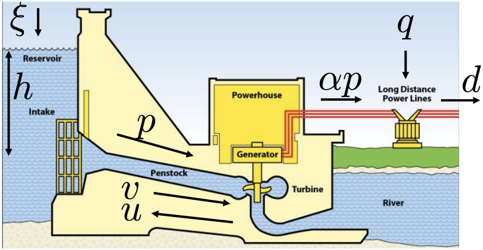
Motivation: Hydrothermal Scheduling



$$
\begin{aligned}
\min \quad & \sum_{t=1}^{T} b_t q_t + c_t q_t + g_t v_t && \} \text{ generation \& spillage costs} \\
\text{s.t.} \quad & h_t = h_{t-1} + \xi_t - p_t + u_t - v_t, \quad \forall t && \} \text{ reservoir balance} \\
& \alpha p_t + q_t = d_t, \quad \forall t && \} \text{ meet power demand} \\
& 0 \leq h_t \leq h^{\max}, \;\; p_t, q_t, v_t, u_t \geq 0, \quad \forall t &&
\end{aligned}
$$

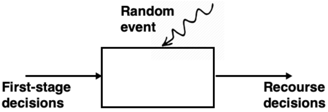
- h_t: amount of water in the reservoir at stage t
- ξ_t: uncertain amount of rainfall at stage t

Outline

1. Data-driven two-stage stochastic optimization

2. Multi-stage stochastic optimization on time series

Prelude: Two-Stage Stochastic Programming



- Traditional two-stage SP: minimize expected system cost *assuming* distribution of random vector Y known

$$\min_{z \in \mathcal{Z}} \mathbb{E}_Y[c(z, Y)]$$

- Sample Average Approximation: given samples $\{y^i\}_{i=1}^n$ of Y

$$\min_{z \in \mathcal{Z}} \mathbb{E}_Y[c(z, Y)] \approx \min_{z \in \mathcal{Z}} \frac{1}{n} \sum_{i=1}^{n} c(z, y^i)$$

- SAA theory: optimal value and solutions converge as $n \to \infty$

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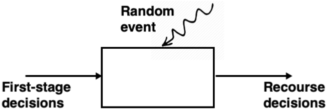

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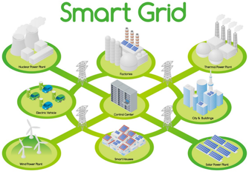
Can we use covariates/features to better predict the random vector Y?

Stochastic Programming with Covariate Information



Power Grid Scheduling

Y: Load; Renewable energy outputs

X: Weather observations; Time/Season

z: Generator scheduling decisions

Production Planning/Scheduling

Y: Product demands; Prices

X: Seasonality; Web search results

z: Production and inventory decisions

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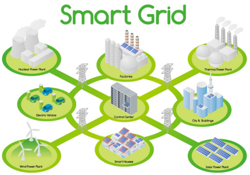

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- Given historical data on uncertain parameters and covariates

$$\mathcal{D}_n := \{(y^i, x^i)\}_{i=1}^n$$

- When making decision z, we observe a *new* covariate $X = x$
- Goal: minimize expected cost given covariate observation x:

$$\min_{z \in \mathcal{Z}} \mathbb{E}\left[c(z, Y) \mid X = x\right]$$

Stochastic Programming with Covariate Information

- Assume we have uncertain parameter and covariate data pairs

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  1. Learn: predict Y given $X = x$
  2. Optimize: integrate learning into optimization (with errors)

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- How to construct data-driven approximation to conditional SP?
  1. Learn: predict Y given $X = x$
  2. Optimize: integrate learning into optimization (with errors)
- Assume $Y = f^*(X) + Q^*(X)\varepsilon$ with X and ε *independent*

Separate Learning and Optimization

1. Use data to train our favorite ML prediction model:

$$\hat{f}_n(\cdot) \in \arg\min_{f(\cdot)\in\mathcal{F}} \sum_{i=1}^{n} \ell(f(x^i), y^i) + \rho(f)$$

2. Given observed covariate $X = x$, use point prediction within deterministic optimization model

$$\min_{z\in\mathcal{Z}} c(z, \hat{f}_n(x))$$

- Modular: separate learning and optimization steps
- Expect to work well if (and likely only if) prediction is accurate
- Does not yield asymptotically consistent solutions

Integrated Learning and Optimization

Approach 1: Modify the learning step[1]

- Change loss function in ML training step to reflect use of prediction within optimization model
- More challenging training problem + less modular

Approach 2: Modify the optimization step[2]

- Change optimization model to reflect uncertainty in prediction

Approach 3: Direct solution learning[3]

- Attempt to directly learn a mapping from x to a solution z
- Handling constraints and large dimensions of z is challenging

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[1] Kao et al. [2009], Donti et al. [2017], Elmachtoub and Grigas [2017]
[2] Ban et al. [2018], Bertsimas and Kallus [2020], Deng and Sen [2022]
[3] Ban and Rudin [2018], Bertsimas and Kallus [2020]

Empirical Residuals-based Sample Average Approximation

Approach (Deng and Sen [2022],Ban et al. [2018],K. et al. [2020a])

1. Use data to train our favorite ML prediction model $\Rightarrow \hat{f}_n, \hat{Q}_n$

$$\hat{f}_n(\cdot) \in \operatorname*{arg\,min}_{f(\cdot)\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n}\|y^i - f(x^i)\|^2$$

Compute *empirical residuals* $\hat{\varepsilon}_n^i := [\hat{Q}_n(x^i)]^{-1}\big(y^i - \hat{f}_n(x^i)\big),\ i \in [n]$

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2. Use $\{\hat{f}_n(x) + \hat{Q}_n(x)\hat{\varepsilon}_n^i\}_{i=1}^n$ as proxy for samples of Y given $X = x$

$$\min_{z\in\mathcal{Z}} \frac{1}{n}\sum_{i=1}^{n} c(z, \hat{f}_n(x) + \hat{Q}_n(x)\hat{\varepsilon}_n^i) \qquad \text{(ER-SAA)}$$

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- DRO extension: K. et al. [2020b]

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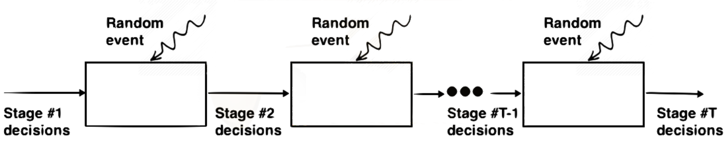
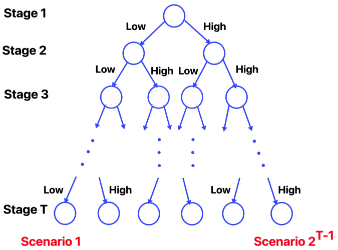
- Convergence conditions and rates: K. et al. [2020a]
- DRO extension: K. et al. [2020b]
- Can we extend approach to multi-stage case, particularly given *a single* historical sequence of data?

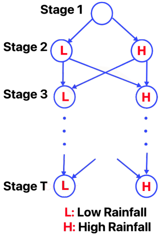
Outline

Multistage Stochastic Optimization



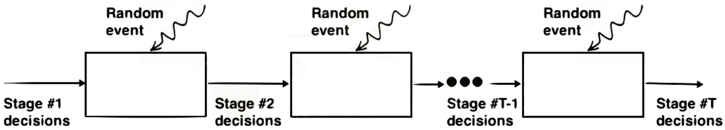
Complexity of multi-stage stochastic programs can grow significantly with the number of stages T!





Stochastic Dual Dynamic Programming (Pereira and Pinto [1991]): Exploit recombining scenario tree structure to limit number of value functions that need to be approximated.

Multistage Stochastic Optimization



- Decision Process: $z_1 \rightsquigarrow \xi_2 \rightsquigarrow z_2 \rightsquigarrow \cdots \xi_T \rightsquigarrow z_T$

At stage t, solve

$$\min_{z_t \in Z_t(z_{t-1},\xi_t)} \begin{array}{c}\text{cost of decisions } z_t\\ \text{in current stage } t\end{array} + \begin{array}{c}\text{expected cost of decisions } z_t\\ \text{in future stages given history } (\xi_1,\ldots,\xi_t)\end{array}$$

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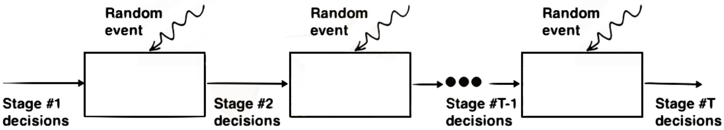

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At stage t, solve

$$\min_{z_t \in Z_t(z_{t-1}, \xi_t)} \begin{matrix}\text{cost of decisions } z_t \\ \text{in current stage } t\end{matrix} + \begin{matrix}\text{expected cost of decisions } z_t \\ \text{in future stages given history } (\xi_1, \dots, \xi_t)\end{matrix}$$

- Assume time series model: $\xi_t = f^*(\xi_{t-1}) + Q^*(\xi_{t-1})\varepsilon_t$
- Goal: Given a *single historical trajectory* of $\{\xi_t\}$

$$\mathcal{D}_n := \{\tilde{\xi}_0, \tilde{\xi}_1, \cdots, \tilde{\xi}_n\}$$

estimate optimal first-stage decision z_1

Related work

Bertsimas et al. [2022]:

- Assume given an *i.i.d. set of historical sample paths*
- Construct RO model with uncertainty sets around sample paths
- Show asymptotic convergence as *number of sample paths grows*
- Solve using decision rule approximations
- Related: Ban et al. [2018], Bertsimas and McCord [2019], Bertsimas et al. [2019]

Silva et al. [2021]:

- Assume *single historical sample path*, fit Hidden Markov Model
- Construct DRO model with ambiguity set for transition prob.
- Solve by adapting Stochastic Dual Dynamic Programming
- No analysis of *convergence to true problem*

Related work and Goals

Guevara et al. [2022]:

- Assume *single historical sample path*
- Fit a linear AR model with prespecified ranges of variation
- Solve finite-state Markovian approximation using SDDP
- No analysis of *convergence to true problem*

Related work and Goals

Guevara et al. [2022]:

- Assume *single historical sample path*
- Fit a linear AR model with prespecified ranges of variation
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- No analysis of *convergence to true problem*

Our goals:

- Use *single historical sample path*
- Construct data-driven approximation that can be solved using Stochastic Dual Dynamic Programming
- Establish convergence as *size of sample path grows* (assuming time series model)

Problem Setup

- Given historical data from a *single trajectory* of $\{\xi_t\}$

$$\mathcal{D}_n := \{\tilde{\xi}^0, \tilde{\xi}^1, \cdots, \tilde{\xi}^n\}$$

- Want to solve

$$V_1(\xi_1) := \min_{z_1 \in Z_1(\xi_1)} f_1(z_1, \xi_1) + \mathbb{E}\left[V_2(z_1, \xi_2) \mid \xi_1\right],$$

where

$$V_t(z_{t-1}, \xi_{[t]}) := \min_{z_t \in Z_t(z_{t-1}, \xi_t)} \overbrace{f_t(z_t, \xi_t)}^{\text{stage } t \text{ cost}} + \overbrace{\mathbb{E}\left[V_{t+1}(z_t, \xi_{[t+1]}) \mid \xi_{[t]}\right]}^{\text{expected cost of future stages}}, \quad t \in [T-1],$$

$$V_T(z_{T-1}, \xi_{[T]}) := \min_{z_T \in Z_T(z_{T-1}, \xi_T)} f_T(z_T, \xi_T).$$

- Assume
  - True model: $\xi_t = f^*(\xi_{t-1}) + Q^*(\xi_{t-1})\varepsilon_t$ with i.i.d. errors $\{\varepsilon_t\}$
  - We know function classes $\mathcal{F}$, $\mathcal{Q}$ such that $f^* \in \mathcal{F}$, $Q^* \in \mathcal{Q}$

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$$\hat{V}_{t,n}^{ER}(z_{t-1}, \xi_t) := \min_{z_t \in Z_t(z_{t-1}, \xi_t)} f_t(z_t, \xi_t) + \frac{1}{n}\sum_{j \in [n]} \hat{V}_{t+1,n}^{ER}(z_t, \hat{f}_n(\xi_t) + \hat{Q}_n(\xi_t)\hat{\varepsilon}_n^i)$$

Tailored convergence analysis required since same empirical errors $\hat{\varepsilon}_n^i$ used for all time stages

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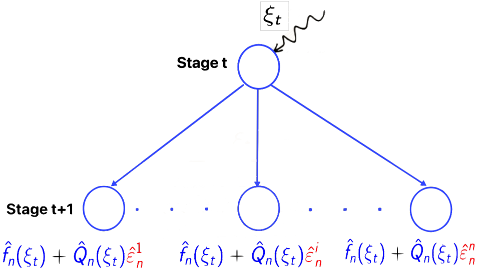

Convergence Theory

Assumptions on the multistage stochastic program:

Assumptions on the ML setup:

Asymptotic optimality

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- Can always take recourse decisions to keep system feasible
- The feasible region Z_t for each stage t is bounded

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Assumptions on the ML setup:

- The functions f^* and Q^* are Lipschitz continuous
- $\hat{f}_n \to f^*$ and $\hat{Q}_n \to Q^*$ uniformly on their domains

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Asymptotic optimality

Under above assumptions, as the historical sample size n increases, any first-stage ER-SAA solution converges to an optimal solution of the true multistage stochastic program

Convergence Theory

Result holds with these weaker assumptions on the ML setup:

- The functions f^*, $\hat{f}_n$, Q^*, and $\hat{Q}_n$ are Lipschitz continuous
- Mean-squared estimation error consistency:

$$\frac{1}{n}\sum_{i\in[n]} \|f^*(\tilde{\xi}^{i-1}) - \hat{f}_n(\tilde{\xi}^{i-1})\|^2 \xrightarrow{p} 0,$$

$$\frac{1}{n}\sum_{i\in[n]} \big\|\big[Q^*(\tilde{\xi}^{i-1})\big]^{-1} - \big[\hat{Q}_n(\tilde{\xi}^{i-1})\big]^{-1}\big\|^2 \xrightarrow{p} 0$$

- For each $t \in [T-1]$:

$$\mathbb{E}_{\varepsilon_t \sim P_n}\left[\|f^*(\xi_t) - \hat{f}_n(\xi_t)\| \,\middle|\, \xi_1\right] \xrightarrow{p} 0,$$

$$\mathbb{E}_{\varepsilon_t \sim P_n}\left[\|Q^*(\xi_t) - \hat{Q}_n(\xi_t)\| \,\middle|\, \xi_1\right] \xrightarrow{p} 0$$

$P_n := \frac{1}{n}\sum_{i\in[n]} \delta_{\tilde{\varepsilon}^i}$ is the true empirical distribution of errors

These assumptions can be readily verified, e.g., for linear vector auto-regressive processes

Rates of Convergence

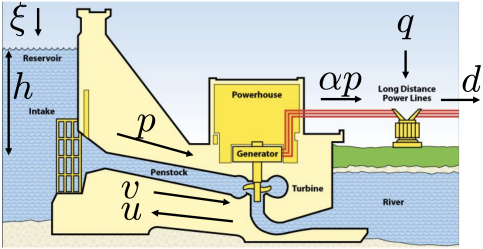
Assume

- The errors $\{\varepsilon_t\}$ are sub-Gaussian
- The true multistage stochastic program satisfies assumptions required for SAA convergence (e.g., Shapiro et al. [2009])
- The regression estimates $\hat{f}_n$ and $\hat{Q}_n$ satisfy large deviation properties

Rates of convergence of regression estimates dictate rates of convergence of ER-SAA solutions

- For parametric time series models, rate of convergence of ER-SAA can equal rate of convergence of classical SAA

Numerical Experiments: Hydrothermal Scheduling

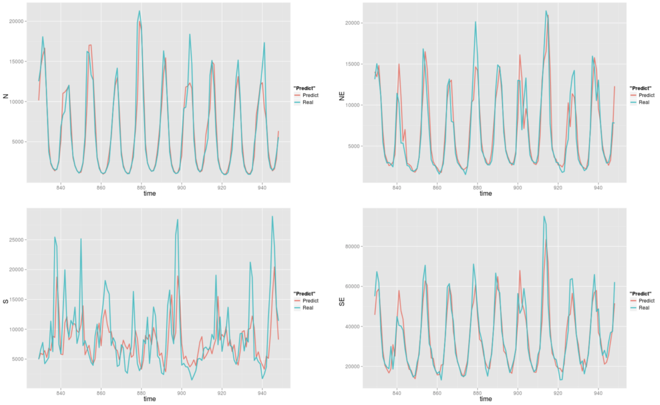


- Decisions z_t: Hydrothermal & natural gas generation, spillage
- Random vector ξ: Amount of rainfall

Assume true time series model for rainfall is of the form

$$\xi_t = (\alpha_t^* + \beta_t^* \xi_{t-1}) \exp(\varepsilon_t),$$

where $\alpha_t^* = \alpha_{t+12}^*, \quad \beta_t^* = \beta_{t+12}^*, \quad \varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \Sigma)$



Good fit to historical data over 8 decades! (Shapiro et al. [2012])

Numerical Experiments: Hydrothermal Scheduling

- Consider the Brazilian interconnected power system with four hydrothermal reservoirs

- Generate a sample trajectory of $\{\xi_t\}$ using time series model

$$\xi_t = (\alpha_t^* + \beta_t^* \xi_{t-1}) \exp(\varepsilon_t),$$

  where $\alpha_t^* = \alpha_{t+12}^*, \quad \beta_t^* = \beta_{t+12}^*, \quad \varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \Sigma)$

- Estimate coefficients $(\hat{\alpha}_t, \hat{\beta}_t)$ such that

$$\hat{\alpha}_t = \hat{\alpha}_{t+12}, \quad \hat{\beta}_t = \hat{\beta}_{t+12}$$

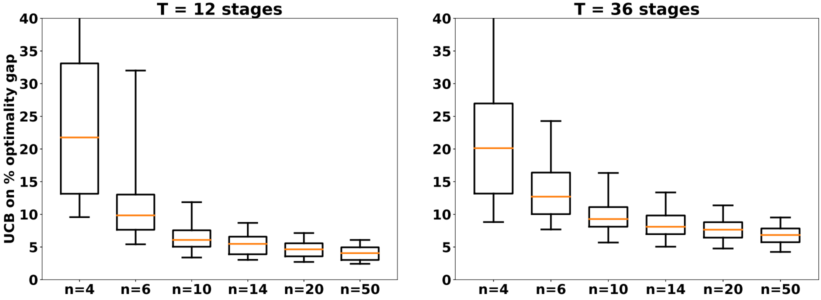
  Use these to estimate samples of the errors ε_t

- Solve the ER-SAA model using `SDDP.jl` [Dowson and Kapelevich, 2021]. Estimate sub-optimality of solutions

Results when the time series model is correctly specified

Estimate true heteroscedastic model: $\quad \xi_t = (\alpha_t^* + \beta_t^* \xi_{t-1}) \exp(\varepsilon_t)$

Lower y-axis value $\implies$ closer to optimal



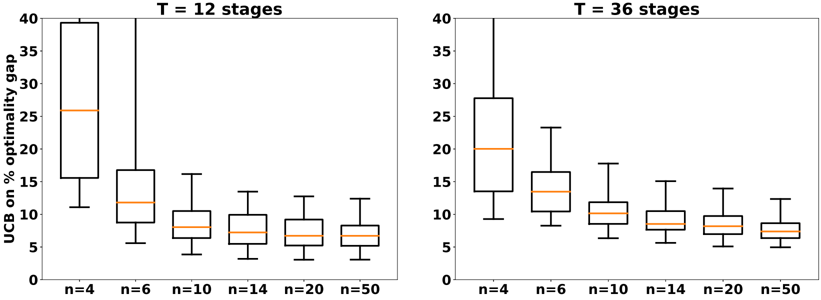
n: number of historical samples *per month*

Boxes: 25, 50, and 75 percentiles of optimality gap estimates;
Whiskers: 5 and 95 percentiles

Results when the time series model is misspecified

Estimate seasonal additive error model: $\xi_t = \alpha_t^* + \beta_t^* \xi_{t-1} + \varepsilon_t$

Lower y-axis value $\implies$ closer to optimal



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Concluding Remarks

ER-SAA: a modular approach to using covariate information in optimization under uncertainty

- Solvable using *Stochastic Dual Dynamic Programming*
- Enables decision-makers to effectively use side information

Future research directions

- Formulations with stochastic constraints, discrete recourse decisions; robust multistage optimization
- Application to energy systems optimization

Try it out for your application!

Questions? `rohitk@alum.mit.edu`

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