A Decomposition Strategy for a Class of Nonconvex Two-Stage Stochastic Programs

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Motivation

- Uncertainty in problem data is a common feature of many real-life problems

Sarawak Gas Production System

- Annual revenue of US $5 billion (4% of Malaysia’s GDP)

Motivation
More Chemical Engineering Applications

- Pharmaceutical capacity planning
- Integrated Water Networks

Motivation
Importance of Addressing Uncertainties

Stochastic pooling problem

Superstructure

![Diagram showing relationships between S1, S2, S3, S4, P1, T1, and T2]
Motivation
Importance of Addressing Uncertainties

Stochastic pooling problem

Superstructure

Deterministic solution

Stochastic solution
Two-Stage Stochastic Programming Framework

Stage 1 decisions
- made before the realization of the uncertainty
- e.g., design decisions

Realization of the uncertainty
- e.g., source qualities

Stage 2 decisions
- made after the realization of the uncertainty
- e.g., operational decisions
Nonconvex Two-Stage Stochastic Programs

\[
\min_{x_1, \ldots, x_s, y, z} \sum_{h=1}^{s} p_h f_h(x_h, y, z) \tag{P}
\]

s.t. \[ g_h(x_h, y, z) \leq 0, \ \forall h \in \{1, \ldots, s\}, \]

\[ x_h \in X_h \subset \{0,1\}^{n_{xb}} \times \mathbb{R}^{n_{xc}}, \ \forall h \in \{1, \ldots, s\}, \]

\[ y \in Y \subset \{0,1\}^{n_y}, \ z \in Z \subset \mathbb{R}^{n_z}. \]

Assumptions: \( f_h, g_h, \ \forall h \in \{1, \ldots, s\}, \) continuous;
\( X_h, \ \forall h \in \{1, \ldots, s\}, \) \( Y, \) and \( Z \) nonempty; and
\( X_h, \ \forall h \in \{1, \ldots, s\}, \) and \( Z \) compact.
Existing Decomposition Approaches

\[
\begin{aligned}
\min_{x_1, \ldots, x_s, y} & \sum_{h=1}^{s} p_h f_h(x_h, y) \\
\text{s.t.} & \quad g_h(x_h, y) \leq 0, \; \forall h \in \{1, \ldots, s\}, \\
& \quad x_h \in X_h \subset \{0,1\}^{n_{xb}} \times \mathbb{R}^{n_x}, \; \forall h \in \{1, \ldots, s\}, \\
& \quad y \in Y \subset \{0,1\}^{n_y}.
\end{aligned}
\]

Nonconvex Generalized Benders Decomposition

\[
\begin{aligned}
\min_{x_1, \ldots, x_s, y_h, z_h} & \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h) \\
\text{s.t.} & \quad g_h(x_h, y_h, z_h) \leq 0, \; \forall h \in \{1, \ldots, s\}, \\
& \quad y_h - y_{h+1} = 0, \; \forall h \in \{1, \ldots, s-1\}, \\
& \quad z_h - z_{h+1} = 0, \; \forall h \in \{1, \ldots, s-1\}, \\
& \quad x_h \in X_h \subset \{0,1\}^{n_{xb}} \times \mathbb{R}^{n_x}, \; \forall h \in \{1, \ldots, s\}, \\
& \quad y_h \in Y \subset \{0,1\}^{n_y}, z_h \in Z \subset \mathbb{R}^{n_z}, \; \forall h \in \{1, \ldots, s\}.
\end{aligned}
\]

Lagrangian Relaxation
Nonconvex Generalized Benders Decomposition

Nonconvex Generalized Benders Decomposition

At present, the NGBD algorithm only converges if the first-stage (complicating) decisions are integers. In this case, finite convergence to a solution within a given tolerance of a global minimum is guaranteed.
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- In this case, finite convergence to a solution within a given tolerance of a global minimum is guaranteed

In practice, only a small fraction of the integer realizations in the set $Y$ are visited by the primal problem

- Strength of relaxations of the nonconvex functions is important
Lagrangian Relaxation

\[
\min_{x_1, \cdots, x_s, \ y_1, \cdots, y_s, \ z_1, \cdots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h)
\]

s.t. \( g_h(x_h, y_h, z_h) \leq 0, \ \forall h \in \{1, \cdots, s\} \),

\( y_h - y_{h+1} = 0, \ \forall h \in \{1, \cdots, s-1\} \),

\( z_h - z_{h+1} = 0, \ \forall h \in \{1, \cdots, s-1\} \),

\( x_h \in X_h \subset \{0, 1\}^{n_{xb}} \times \mathbb{R}^{n_{xc}}, \ \forall h \in \{1, \cdots, s\} \),

\( y_h \in Y \subset \{0, 1\}^{n_y}, \ z_h \in Z \subset \mathbb{R}^{n_z}, \ \forall h \in \{1, \cdots, s\} \).

- The non-anticipativity constraints are hard constraints which link the different scenario problems
Dualizing the non-anticipativity constraints provides a valid lower bounding problem ...

\[
\begin{align*}
\max_{\alpha_1, \ldots, \alpha_{s-1},} \quad \min_{x_1, \ldots, x_s, \ y_1, \ldots, y_s, \ z_1, \ldots, z_s} \quad & \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h) + \sum_{h=1}^{s-1} \alpha_h^T (y_h - y_{h+1}) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \quad (LRP) \\
& \text{s.t. } g_h(x_h, y_h, z_h) \leq 0, \quad \forall h \in \{1, \ldots, s\}, \\
& \quad x_h \in X_h, \ y_h \in Y, \ z_h \in Z, \quad \forall h \in \{1, \ldots, s\}. 
\end{align*}
\]

\(\alpha_h, \beta_h\) are Lagrange multipliers.
Lagrangian Relaxation
Lower Bounding Problem

\[
\max_{\alpha_1, \ldots, \alpha_{s-1}, \beta_1, \ldots, \beta_{s-1}} \min_{x_1, \ldots, x_s, y_1, \ldots, y_s, z_1, \ldots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y_h, z_h) + \sum_{h=1}^{s-1} \alpha_h^T (y_h - y_{h+1}) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \quad \text{(LRP)}
\]

s.t. \( g_h(x_h, y_h, z_h) \leq 0, \ \forall h \in \{1, \ldots, s\}, \)

\[ x_h \in X_h, \ y_h \in Y, \ z_h \in Z, \ \forall h \in \{1, \ldots, s\}. \]

- Dualizing the non-anticipativity constraints provides a valid lower bounding problem ...
  
  ... the inner minimization of which is decomposable
A branch and bound algorithm is employed to guarantee convergence

- Sufficient to branch on the complicating variables \((y, z)\) to converge
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- Sufficient to branch on the complicating variables \((y, z)\) to converge

Branching rule
- Branch on the complicating variable with the maximum dispersion
- An occasional bisection is performed to guarantee convergence
A branch and bound algorithm is employed to guarantee convergence
  Sufficient to branch on the complicating variables \((y, z)\) to converge

Branching rule
  - Branch on the complicating variable with the maximum dispersion
  - An occasional bisection is performed to guarantee convergence

The conventional Lagrangian relaxation algorithm may take a long time to converge
  - Requires the solution of several nonconvex MINLPs to obtain lower bounds
  - Multiple iterations of an algorithm applied to the dual may be required to generate tight lower bounds
**Improved Lagrangian Relaxation**

- **Observation:** The inner minimization of the dual problem can be solved using NGBD if the non-anticipativity constraints of only the continuous complicating variables are relaxed

\[
\begin{align*}
\max_{\alpha_1, \ldots, \alpha_{s-1}, \beta_1, \ldots, \beta_{s-1}, \ x_1, \ldots, x_s, \ y_1, \ldots, y_s, \ z_1, \ldots, z_s} & \min \sum_{h=1}^{s} p_h f_h (x_h, y_h, z_h) + \sum_{h=1}^{s-1} \alpha_h^T (y_h - y_{h+1}) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \\
\text{s.t. } g_h (x_h, y_h, z_h) & \leq 0, \ \forall h \in \{1, \ldots, s\}, \\
x_h & \in X_h, \ y_h \in Y, \ z_h \in Z, \ \forall h \in \{1, \ldots, s\}.
\end{align*}
\]
Improved Lagrangian Relaxation

\[
\begin{align*}
\max_{\beta_1, \ldots, \beta_{s-1}} & \quad \min_{x_1, \ldots, x_s, y, z_1, \ldots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \\
\text{s.t.} & \quad g_h(x_h, y, z_h) \leq 0, \forall h \in \{1, \ldots, s\}, \\
& \quad x_h \in X_h, z_h \in Z, \forall h \in \{1, \ldots, s\}, \\
& \quad y \in Y.
\end{align*}
\]
Improved Lagrangian Relaxation

\[
\begin{align*}
\max_{\beta_1, \ldots, \beta_{s-1}} \quad & \min_{x_1, \ldots, x_s, \atop y, z_1, \ldots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \\
\text{s.t.} \quad & g_h(x_h, y, z_h) \leq 0, \quad \forall h \in \{1, \ldots, s\}, \\
& x_h \in X_h, \quad z_h \in Z, \quad \forall h \in \{1, \ldots, s\}, \\
& y \in Y.
\end{align*}
\]

- The inner minimization of Problem (LRP-NGBD) is not decomposable, but can be solved in a decomposable manner using NGBD
Improved Lagrangian Relaxation

\[
\begin{align*}
\max_{\beta_1, \ldots, \beta_{s-1}} \quad & \min_{x_1, \ldots, x_s, y, z_1, \ldots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \beta_h^T (z_h - z_{h+1}) \\
\text{s.t.} & \quad g_h(x_h, y, z_h) \leq 0, \quad \forall h \in \{1, \ldots, s\}, \\
& \quad x_h \in X_h, \quad z_h \in Z, \quad \forall h \in \{1, \ldots, s\}, \\
& \quad y \in Y.
\end{align*}
\]

- The inner minimization of Problem (LRP-NGBD) is not decomposable, but can be solved in a decomposable manner using NGBD

- The above lower bounding problem provides tighter lower bounds than Problem (LRP)

- Sufficient to branch on the continuous complicating variables to converge
Tight bounds on the continuous complicating variables $z$ may be required for good empirical convergence.

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Suppose UBD is the current best upper bound for Problem (P), and $z^i \in [z_{i,lo}, z_{i,up}]$. If, for some $(\bar{z}^i, \bar{\beta})$, the optimal solution of the following lower bounding problem lies above UBD, then $z^i \in [\bar{z}^i, z_{i,up}]$ is a valid tightening:

$$\min_{x_1, \ldots, x_s, y, z_1, \ldots, z_s} \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \bar{\beta}_h^T (z_h - z_{h+1})$$

subject to

$$g_h(x_h, y, z_h) \leq 0, \quad \forall h \in \{1, \ldots, s\},$$

$$x_h \in X_h, \quad z_h \in Z \cap \{z : z^i \leq \bar{z}^i\}, \quad \forall h \in \{1, \ldots, s\},$$

$$y \in Y.$$
Tight bounds on the continuous complicating variables $z$ may be required for good empirical convergence.

Suppose UBD is the current best upper bound for Problem (P), and $z^i \in [z^{i,lo}, z^{i,up}]$. If, for some $(\bar{z}^i, \bar{\beta})$, the optimal solution of the following lower bounding problem lies above UBD, then $z^i \in [\bar{z}^i, z^{i,up}]$ is a valid tightening.

\[
\min \sum_{h=1}^{s} p_h f_h(x_h, y, z_h) + \sum_{h=1}^{s-1} \bar{\beta}_h^T (z_h - z_{h+1}) \\
\text{s.t. } g_h(x_h, y, z_h) \leq 0, \ \forall h \in \{1, \ldots, s\}, \\
\quad x_h \in X_h, \ z_h \in Z \cap \{z : z^i \leq \bar{z}^i\}, \ \forall h \in \{1, \ldots, s\}, \\
\quad y \in Y.
\]

Nonconvex MINLP

Can be solved using NGBD!

Multiple ABT iterations are carried out on a per-variable basis.

Solution of the lower bounding problem for ABT can be terminated if:
- the lower bound for the lower bounding problem, obtained during the NGBD algorithm, is larger than the current upper bound (fewer primal problems solved)
- a feasible solution for the lower bounding problem which is smaller than the current upper bound is found
ABT requires a good upper bound to be able to effectively tighten the bounds of the continuous complicating variables.
ABT requires a good upper bound to be able to effectively tighten the bounds of the continuous complicating variables.

Good upper bounds can be generated by solving Problem (P) using DICOPT, or by restricting the binary variables and solving the resulting problem using local solvers such as CONOPT.

Local solvers which utilize the near-decomposable structure of Problem (P), through techniques such as Schur complements, can be employed.
Improved Lagrangian Relaxation
Upper Bounds

- ABT requires a good upper bound to be able to effectively tighten the bounds of the continuous complicating variables

- Good upper bounds can be generated by solving Problem (P) using DICOPT, or by restricting the binary variables and solving the resulting problem using local solvers such as CONOPT
  - Local solvers which utilize the near-decomposable structure of Problem (P), through techniques such as Schur complements, can be employed

- Upper bounds can also be generated by attempting to solve Problem (P) using NGBD, or by restricting the continuous complicating variables and solving the resulting problem using NGBD
Improved Lagrangian Relaxation
Flowchart

- Initialization
  - Variable Bounds, UBD, LBD, Tolerances
  - Bounds Tightening
  - Feasible?
    - Yes: Update Variable Bounds
    - No: Update UBD
  - Upper Bounding Problem
  - Feasible?
    - Yes: Update Variable Bounds
    - No: Update LBD
  - Dual algorithm (lower bound)
  - Converged?
    - Yes: Terminate
    - No: Partition Node

- NGBD
  - Variable Bounds, Lower Bound
  - Lagrange Multiplier, Lower Bound, Subgradient

- Terminate
Computational Studies
Implementation Details

◆ Platform
  - CPU 3.07 GHz, Memory 12.0 GB, VMWare Linux Workstation on Windows 7 Desktop, GAMS 24.2, GCC 4.8.1, GFortran 4.8.1

◆ Solvers
  - LP and MILP solver: CPLEX
  - Global NLP solver: ANTIGONE
  - Local NLP solver: CONOPT
  - Upper bound solver: DICOPT
  - Bundle solver: MPBNGC 2.0

◆ Methods for comparison
  - ANTIGONE, BARON – State-of-the-art global optimization solvers
  - Conventional Lagrangian relaxation algorithm
  - Improved Lagrangian relaxation algorithm

◆ Relative and absolute tolerance: $10^{-3}$
Computational Study

Case Study 1: Stochastic Pooling Problem

16 binary complicating variables,
4 continuous complicating variables,
13s continuous recourse variables
16s bilinear terms
(s represents the number of scenarios).
Computational Study
Case Study 2: Integrated Water Network

15 binary complicating variables,
20 continuous complicating variables,
94s continuous recourse variables
242s bilinear terms

(s represents the number of scenarios).
Conclusions and future work

- Nonconvex generalized Benders decomposition can be used in conjunction with bounds tightening techniques to improve the performance of the Lagrangian relaxation algorithm for general nonconvex two-stage stochastic programs.

- Develop decomposition techniques to obtain good upper bounds.

- Look at efficient ways to solve the dual problem.

- Extend the algorithm to multi-stage problems.
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