A stochastic approximation method for chance-constrained nonlinear programs

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Joint work with Jim Luedtke
Outline

1. Introduction
   - Formulation
   - Central hypothesis
   - Main idea

2. Related approaches

3. Proposed Approach

4. Computational results

5. Summary and Open questions
Formulation

\[ \nu^* := \min_{x \in X} f(x) \quad \text{(CCP)} \]

s.t. \( \mathbb{P}\{g(x, \xi) \leq 0\} \geq 1 - \alpha. \)

• Assume:
  - \( X \subset \mathbb{R}^n \) is nonempty, compact, and convex
  - \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous and quasiconvex
  - \( g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m \) is continuously differentiable
  - \( \xi \sim \mathcal{P} \) is continuous with support \( \Xi \subset \mathbb{R}^d \)
  - Other relatively mild technical assumptions . . .
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  ▶ Other relatively mild technical assumptions . . .

• Can model joint chance constraints, deterministic nonconvex constraints, and some recourse structures
In many cases, decision makers are interested in generating the efficient frontier of optimal objective function value ($\nu^*$) versus risk level ($\alpha$) rather than simply solving (CCP) for a single prespecified risk level so that they can make a more informed decision.

— Rengarajan and Morton [2009], Luedtke [2014]
Central hypothesis

Main idea #1

In many cases, decision makers are interested in generating the efficient frontier of optimal objective function value ($\nu^*$) versus risk level ($\alpha$) rather than simply solving (CCP) for a single prespecified risk level so that they can make a more informed decision.

— Rengarajan and Morton [2009], Luedtke [2014]

• Efficient frontier of (CCP) can be recovered by solving the following stochastic optimization problem:

\[
\min_{x \in X} \mathbb{P}\{g(x, \xi) \not\leq 0\} \equiv \min_{x \in X} \mathbb{E}[\max\{1 \cdot [g(x, \xi)]\}] \quad \text{(SP)}
\]

\[\text{s.t. } f(x) \leq \nu.\]

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▶ $1 \cdot [\cdot]$ denotes the l.s.c. step function
Main idea #2

- Approximate the efficient frontier of (CCP) using:

\[
\min_{x \in X} \mathbb{E} \left[ \max \left[ 1 \left[ g(x, \xi) \right] \right] \right] \approx \min_{x \in X} \mathbb{E} \left[ \max \left[ \phi_k (g(x, \xi)) \right] \right] \quad \text{(APP}_k \text{)}
\]

s.t. \( f(x) \leq \nu \).

\( \{ \phi_k \} \) is a ‘convergent’ seq. of smooth approx. of the step function.
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s.t. \( f(x) \leq \nu. \)  

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- (APP\(_k\)) can be solved using stochastic gradient methods.
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Outline

1. Introduction

2. Related approaches
   - Scenario approximation
   - Smoothing-based approaches
   - Motivation for proposed approach

3. Proposed Approach

4. Computational results

5. Summary and Open questions
Scenario approximation

- Sample $N$ random realizations of $\xi$ from $\mathcal{P}$
- Solve the following scenario problem to local optimality using a cutting-plane approach:

$$
\hat{x}_N \in \arg \min_{x \in X} f(x) \\
\text{s.t. } g(x, \xi_j) \leq 0, \quad \forall j \in \{1, \cdots, N\}.
$$

- Estimate the risk level $\hat{\alpha}_N := \mathbb{P}\{g(\hat{x}_N, \xi) \not\leq 0\}$ using an independent Monte Carlo sample to determine the point $(\hat{\alpha}_N, f(\hat{x}_N))$ on the approximation of the efficient frontier

- References: Calafiore and Campi [2005], Campi and Garatti [2011]
Existing smoothing-based approaches

• Approximate the solution of (CCP) using:

\[
\min_{x \in X} f(x) \approx \min_{x \in X} f(x)
\]

s.t. \( \mathbb{E} \left[ \max \left[ 1 \left[ g(x, \xi) \right] \right] \right] \leq \alpha \).

S.t. \( \mathbb{E} \left[ \max \left[ \phi_k (g(x, \xi)) \right] \right] \leq \alpha \).

where \( \{\phi_k\} \) is a ‘convergent’ sequence of conservative smooth approximations of the step function.

• Solution of each approximation yields a feasible solution to the original chance-constrained program.
Existing smoothing-based approaches

- Approximate the solution of (CCP) using:

\[
\min_{x \in X} f(x) \approx \min_{x \in X} f(x)
\]

\[
\text{s.t. } \mathbb{E} [\max [1 \cdot g(x, \xi)]] \leq \alpha.
\]

where \( \{\phi_k\} \) is a ‘convergent’ sequence of conservative smooth approximations of the step function.

- Solution of each approximation yields a feasible solution to the original chance-constrained program

- Currently, the only known approach for solving optimization problems with nonconvex expectation constraints is via sample average approximation

- References: Hong et al. [2011], Shan et al. [2014, 2016], Geletu et al. [2017], Cao and Zavala [2017], Adam et al. [2018]
### Pros and cons of exterior sampling approaches

<table>
<thead>
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<th>Pros</th>
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<tbody>
<tr>
<td>▶ Can use off-the-shelf solvers</td>
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<td>▶ Possess strong theoretical guarantees</td>
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<th>Cons</th>
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<td>▶ May find spurious local optima</td>
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<td>▶ May need to solve large-scale NLPs</td>
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- We propose the first interior sampling approach for solving chance-constrained problems
Sampling may lead to spurious local minima

- Consider the following chance constraint [Curtis et al., 2018]:

\[
P \left\{ 0.25x_1^4 - \frac{1}{3}x_1^3 - x_1^2 + 0.2x_1 - 19.5 + \xi_1x_1 + \xi_1\xi_0 \leq x_2 \right\} \geq 0.95,
\]

where \( \xi_1 \sim U(-3, 3) \) and \( \xi_0 \sim U(-12, 12) \) are independent.
Outline

1 Introduction

2 Related approaches

3 Proposed Approach
   Main theoretical results
   Approximating the efficient frontier

4 Computational results

5 Summary and Open questions
Main theoretical results

- Let \( \{\phi_k\} \) be a ‘convergent’ sequence of smooth approximations of the step function.

\[
\begin{align*}
\min_{x \in X} & \mathbb{E} \left[ \max \left[ 1 \left[ g(x, \xi) \right] \right] \right] \quad (SP) \\
\min_{x \in X} & \mathbb{E} \left[ \max \left[ \phi_k (g(x, \xi)) \right] \right] \quad (APP_k)
\end{align*}
\]
\[
s.t. \ f(x) \leq \nu. \\
s.t. \ f(x) \leq \nu.
\]

**Theorem**

*Every limit point of a sequence of global solutions to the approximations (APP\(_k\)) is a global solution to Problem (SP)*

**Tentative Theorem**

*Every limit point of a sequence of stationary solutions to the approximations (APP\(_k\)) is a stationary solution to Problem (SP)*
Proposal for approximating the efficient frontier

- **Input:** initial guess $\hat{x}^0 \in X$, sequence of objective bounds $\{\bar{\nu}^k\}$ (determined by solving a scenario approximation problem), and lower bound on risk level $\alpha_{\text{low}} \in (0, 1)$
- **Output:** pairs $\{(\bar{\nu}^i, \bar{\alpha}^i)\}$ of objective values and risk levels used to approximate the efficient frontier (+ solutions $\{\bar{x}^i\}$)
Proposal for approximating the efficient frontier

- **Input:** initial guess $\hat{x}^0 \in X$, sequence of objective bounds $\{\bar{\nu}^k\}$ (determined by solving a scenario approximation problem), and lower bound on risk level $\alpha_{low} \in (0, 1)$

- **Output:** pairs $\{({\bar{\nu}}^i, \bar{\alpha}^i)\}$ of objective values and risk levels used to approximate the efficient frontier (+ solutions $\{\bar{x}^i\}$)

- **Solve:** sequence of problems (APP$_k$) with objective bound $\bar{\nu}^k$ using the projected stochastic subgradient method [Davis and Drusvyatskiy, 2018]

- **Estimate:** risk level $\bar{\alpha}^k$ of the best found solution using an independent Monte Carlo sample to determine the point $({\bar{\nu}}^k, \bar{\alpha}^k)$ on our approximation of the efficient frontier

- **Terminate:** when $\bar{\alpha}^k \leq \alpha_{low}$
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3. Proposed Approach

4. Computational results
   - Portfolio optimization
   - Resource planning

5. Summary and Open questions
Portfolio optimization [Ben-Tal et al., 2009]

\[
\max_{t, \mathbf{x} \in \mathbf{X}} t \\
\text{s.t.} \quad \mathbb{P} \left\{ \xi^T \mathbf{x} \geq t \right\} \geq 1 - \alpha,
\]

where \( \mathbf{X} := \left\{ \mathbf{x} \in \mathbb{R}_{+}^{1000} : \sum_i x_i = 1 \right\} \), \( \xi_i \) are independent Normal.
Resource planning [Luedtke, 2014]

\[
\begin{align*}
\min_{x \in \mathbb{R}^{20}_+} & \quad c^T x \\
\text{s.t.} & \quad \mathbb{P}\{x \in R(\lambda, \rho)\} \geq 1 - \alpha,
\end{align*}
\]

where

\[
R(\lambda, \rho) = \left\{ x \in \mathbb{R}^{20}_+ : \exists y \in \mathbb{R}^{20 \times 30}_+ \text{ s.t. } \sum_{j=1}^{30} y_{ij} \leq \rho_i x_i^2, \quad \forall i \in \{1, \ldots, 20\}, \right. \\
& \quad \left. \sum_{i=1}^{20} \mu_{ij} y_{ij} \geq \lambda_j, \quad \forall j \in \{1, \ldots, 30\} \right\}
\]

- $x_i$: quantity of resource $i$, \hspace{1em} $c_i$: unit cost of resource $i$
- $y_{ij}$: amount of resource $i$ allocated to customer type $j$
- $\rho_i \in (0, 1]$: random yield of resource $i$
- $\lambda_j \geq 0$: random demand of customer type $j$
- $\mu_{ij} \geq 0$: service rate of resource $i$ for customer type $j$
Resource planning

![Graph showing resource planning data](image)

- IPOPT scenario approximation
- SCIP scenario approximation
- IPOPT-based initialization
- Stochastic approximation with IPOPT init
- SCIP-based initialization
- Stochastic approximation with SCIP init

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Summary and Open questions

• Proposed a stochastic approximation approach for generating the efficient frontier of chance-constrained NLPs
• Computational results indicate that our proposal consistently determines better approximations of the efficient frontier than existing approaches in reasonable computation times

• Open questions
  • Extension to multiple sets of joint chance constraints
  • Handle deterministic nonconvex constraints more naturally
  • Theory for randomized constraint projection techniques to reduce effort spent on projections
  • Extension to distributionally robust chance constraints
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